ON CONVECTIVE STABILITY IN THE PRESENCE OF PERIODICALLY VARYING PARAMETER

PMM Vol.34, №3, 1970, pp.470-480
G. Z. GERSHUNI, E. M. ZHUKHOVITSKII and Iu. S. IURKOV
(Perm')
(Received December 21, 1969)

The problem of initiation of convection in the presence of a parameter which varies periodically with time is of interest because parametric interaction has a significant influence on stability. The two most natural methods of parametric influence on convective stability are: modulation of the equilibrium temperature gradient and modulation of the external force field. Modulation of the temperature gradient can be achieved by means of periodic variation with time of the temperature on the boundaries of the cavity containing the fluid. Modulation of the external force field (gravity field) asises in the presence of vertical oscillations of the fluid.

These mechanisms of parameteric interaction, generally speaking, are different. Because of the thermal skin effect, the periodic variation of temperature with time on the boundaries of the cavity results in the modulation of the mass (convective) force, only in some layer of thickness of which decreases with increasing frequency. In the case of vertical oscillations of the fluid filled cavity, on the other hand, modualtion of the convective force is realized (in the incompressible fluid) uniformly throughout the entire volume. However, this distinction disappears at relatively low frequencies when the thickness of the thermal skin layer is sufficiently large in comparison to the characteristic linear dimension of the cavity. In this limiting case both methods of parametric excitation turn out to be essentially equivalent.

The effect of periodic variation of a parameter on the initiation of convection was studied in papers [1–6]. In paper [1] the stability of a plane horizontal layer of fluid with free boundaries was studied in the case of periodic modulation of the vertical temperature gradient. The region of low frequencies where it is possible to neglect the thermal skin effect was examined. For the case of rectangular wave modulation the exact solution was found for the equations of small perturbations, and the stability boundaries were determined. In papers [2, 5] the effect of temperature gradient modulation was investigated under conditions where the presence of the thermal skin effect is significant. In paper [2] the problem of initiation of convection in a deep basin, the surface temperature of which varied with time according to the harmonic law, was examined. With the aid of an integral method an estimate of the lower stability boundary was obtained. In [5] the small parameter method was applied to the investigation of stability of a plane horizontal layer with periodic temperature variation on the boundaries. Assuming small amplitudes of temperature modulation the critical Rayleigh numbers were found for different schedules of temperature variation on the free boundaries of the layer. In papers [3, 4] the effect of high-frequency vertical vibrations on convective stability of a plane horizontal fluid layer was studied. By means of the method of averaging, the dependence of the critical Rayleigh number on the vibrational parameter was determined. Finally, in paper [6] the structure of convective motion in the super-critical region was studied for the case of vertical oscillations. The study was conducted on the basis of numerical solution of nonlinear equations of convection.

In the present paper the study started in [1] is continued. The effect of parametric influence (modulation of the vertical temperature gradient or the gravity field) on the stability of equilibrium in a plane horizontal layer with free and rigid boundaries is examined. The case of a vertical circular cylinder is also treated. By means of the method of Kantorovich the system of equations for the perturbations is reduced to a system of ordinary equations for amplitudes which depend on time. Periodic solutions of these equations for the case of sinusoidal modulation are found numerically by the Runge-Kutta method on an electronic computer.

1. Modulation of the vertical temperature gradient. At first let us examine by means of periodic modulation of the temperature gradient the parametric influence on the stability of the equilibrium. We shall have in mind modulations of low frequency where the thermal skin effect may be neglected. In this case the equilibrium temperature gradient is homogeneous over the volume of the fluid and is modulated near an average value A_0 with a frequency ω_0 and an amplitude a_0.

$$\nabla T_0 = -(A_0 + a_0 \sin \omega_0 t)\gamma = -A_0\,(1 + \eta \sin \omega_0 t)\,\gamma \quad (\eta = a_0 / A_0) \tag{1.1}$$

Here γ is the unit vector pointed vertically up.

Let us introduce as units of distance, time, velocity, temperature and pressure L, $L^2/\sqrt{\nu\chi}$, χ/L, $A_0 L$ and $\rho\nu\chi/L^2$ (L is the characteristic dimension of the cavity, ν and χ are coefficients of kinematic viscosity and thermal conductivity), respectively. Then in nondimensional form the equations of small perturbations of equilibrium can be written as

$$\frac{1}{\sqrt{P}}\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \Delta \mathbf{v} + R_f T \gamma \tag{1.2}$$

$$\sqrt{P}\,\frac{\partial T}{\partial t} - (1 + \eta \sin \Omega t)(\mathbf{v}\gamma) = \Delta T, \qquad \operatorname{div} \mathbf{v} = 0 \tag{1.3}$$

$$R_f = \frac{g\beta A_0 L^4}{\nu\chi}, \qquad P = \frac{\nu}{\chi}, \qquad \Omega = \frac{L^2}{\sqrt{\nu\chi}}\omega_0 \tag{1.4}$$

Here R_f and P are the Rayleigh and Prandtl numbers, Ω is the nondimensional frequency of modulation.

Turning to the examination of the plane horizontal fluid layer, we select the origin of coordinates on the lower plane and orient the z-axis vertically up. The x-axis and y-axis are placed horizontally. Eliminating the velocity components v_x and v_y and also the pressure p and introducing normal perturbations

$$v_z = v\,(z, t) \exp i\,(k_1 x + k_2 y), \quad T = \theta\,(z, t) \exp i\,(k_1 x + k_2 y) \tag{1.5}$$

we obtain from (1.2), (1.3) a system of amplitude equations (the prime designates differentiation with respect to z)

$$\frac{1}{\sqrt{P}}\frac{\partial}{\partial t}(v'' - k^2 v) = (v^{\mathrm{IV}} - 2k^2 v'' + k^4 v) - k^2 R_f \theta \tag{1.6}$$

$$\sqrt{P}\,\frac{\partial \theta}{\partial t} - (1 + \eta \sin \Omega t)\, v = \theta'' - k^2\theta \qquad (k^2 = k_1^2 + k_2^2) \tag{1.7}$$

Just as in the absence of modulation, the simplest case for analysis is the case with free boundaries. Then on the boundaries of the layer

$$v = v'' = 0,\ \theta = 0 \quad \text{for } z = 0 \quad \text{and } z = 1 \tag{1.8}$$

(the thickness of the layer h is taken as the characteristic dimension L).

The problem (1.6)–(1.8) has a solution of the form

$$v\,(z,\ t) = a\,(t) \sin \pi z,\ \theta\,(z,\ t) = b\,(t) \sin \pi z \tag{1.9}$$

which corresponds to the fundamental level of instability. The amplitudes $a\ (t)$ and $b\ (t)$ satisfy the system of ordinary equations

$$\begin{gathered} \frac{\varkappa^2}{\sqrt{P}} a^{\cdot} + \varkappa^4 a = k^2 R_f b \qquad (\varkappa^2 = k^2 + \pi^2) \\ \sqrt{P}\, b^{\cdot} + \varkappa^2 b = (1 + \eta \sin \Omega t)\, a \end{gathered} \tag{1.10}$$

Substitution of

$$a = f_1,\ b = l f_2,\ t = m\tau \qquad (l = 1 / \varkappa^2 \sqrt{P},\ m = 1 / \varkappa^2) \tag{1.11}$$

allows to reduce the system (1.10) to the "canonical" form

$$f_1^{\cdot} + n f_1 = R f_2, \qquad f_2^{\cdot} + \frac{1}{n} f_2 = (1 + \eta \sin \omega\tau)\, f_1 \tag{1.12}$$

$$\omega = m\Omega = \frac{mL^2}{\sqrt{\nu\chi}} \omega_0,\ n = \sqrt{P},\ R = \frac{R_f}{R_0},\ R_0 = \frac{\varkappa^6}{k^2} = \frac{(k^2 + \pi^2)^3}{k^2} \tag{1.13}$$

Here the dot indicates derivatives with respect to a new nondimensional time τ, ω is the new nondimensional frequency, η is the amplitude of modulation; the parameter n plays the role of the friction coefficent, and R is the "reduced" Rayleigh number (R_0 is the critical Rayleigh number in the absence of modulation).

The system (1.12) is reduced to an equation of the second order with periodic coefficients

$$f_1^{\cdot\cdot} + \left(n + \frac{1}{n}\right) f_1^{\cdot} + [1 - R\,(1 + \eta \sin \omega\tau)]\, f_1 = 0 \tag{1.14}$$

Equation (1.14) was obtained in [1]. In the same paper regions of stability and instability were found for the case of rectengular modulation.

The case with rigid boundaries of the layer is more complicated when the boundary conditions have the form

$$v = v' = 0,\ \theta = 0 \quad \text{for } z = 0 \quad \text{and } z = 1 \tag{1.15}$$

For the reduction of the problem to a system of ordinary equations we can apply the method of Kantorovich, representing v and θ in the form of expansions

$$v\,(z,\ t) = \sum_i a_i\,(t)\, F_i\,(z), \qquad \theta\,(z,\ t) = \sum_i b_i\,(t)\, \Phi_i\,(z) \tag{1.16}$$

where $F_i\ (z)$ and $\Phi_i\ (z)$ are systems of base coordinate functions which satisfy the boundary conditions (1.15). Substituting (1.16) into system (1.6), (1.7), multiplying by F_i and Φ_i respectively, and integrating with respect to z, we obtain a system of ordinary first order equations with periodic coefficients for amplitudes $a_i\ (t)$ and $b_i\ (t)$. Limiting ourselves to the first approximation, we write

$$v\,(z,\ t) = a\,(t) F\,(z), \qquad \theta\,(z,\ t) = b\,(t) \Phi(z) \tag{1.17}$$

$$F\,(z) = z^2\,(1 - z)^2,\quad \Phi\,(z) = z\,(1 - z)\,(1 + z - z^2) \tag{1.18}$$

In the selection of the approximation $\Phi(z)$ the additional condition $\theta''\,(0) = \theta''\,(1) = 0$, which arises from (1.7), was taken into account.

For amplitudes $a(t)$ and $b(t)$ a system is obtained which differs from (1.10) only by coefficients. This system can be reduced to the form (1.12) through the substitution (1.11) if the following choices are made

$$m = \left[\frac{31(12+k^2)}{(504+24k^2+k^4)(306+31k^2)}\right]^{1/2}, \qquad l = \frac{11}{62}\frac{m}{\sqrt{P}} \tag{1.19}$$

In the case of a layer with rigid boundaries the parameters entering into the system (1.12) are

$$n = \left[\frac{31(504+24k^2+k^4)}{(12+k^2)(306+31k^2)}P\right]^{1/2}$$

$$R = \frac{R_f}{R_0}, \qquad R_0 = \frac{4}{121k^2}(504+24k^2+k^4)(306+31k^2) \tag{1.20}$$

Here R_0 is the approximated value of the critical Rayleigh number in the absence of modulation for a layer with rigid boundaries (the minimum value corresponds to $k_m =$ $= 3.12$ and is equal to 1719; the derivation from the exact value of 1708 is 0.6%).

In this manner the determination of stability boundaries in the presence of modulation of the equilibrium temperature gradient in the case of a layer with free or rigid boundaries is reduced to finding periodic solutions of system (1.12).

2. Modulation of the gravity field. We turn now to the examination of another method of parametric influence on convective stability. Let the fluid filled cavity execute vertical harmonic oscillations. In the frame of reference connected with the cavity it is necessary in the equations of motion to replace the acceleration due to the force of gravity g by $g(1+\eta\sin\omega_0 t)$, where $\eta = \omega_0^2 b_0/g$ is the nondimensional modulation parameter, b_0 is the amplitude of displacement.

Retaining the earlier selected units, we write the equations of perturbations in the nondimensional form

$$\frac{1}{\sqrt{P}}\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \Delta\mathbf{v} + R_f(1+\eta\sin\Omega t)T\boldsymbol{\gamma} \tag{2.1}$$

$$\sqrt{P}\frac{\partial T}{\partial t} - (\mathbf{v}\boldsymbol{\gamma}) = \Delta T, \qquad \operatorname{div}\mathbf{v} = 0 \tag{2.2}$$

Parameters R_f, P and Ω were determined earlier in (1.4). The system (2.1), (2.2) differs from the analogous system (1.2), (1.3), which was obtained in the case of temperature gradient modulation, in that the periodic coefficient $(1+\eta\sin\Omega t)$ now does not enter into the thermal conductivity equation, but into the equation of motion in the form of a coefficient associated with the lifting force. The system of amplitude equations for the plane layer differs from the corresponding system (1.6), (1.7) in exactly the same manner. The substitution of (1.9) for the case of a layer with free boundaries now gives a system for amplitudes $a(t)$ and $b(t)$

$$\frac{\varkappa^2}{\sqrt{P}}a^{\cdot} + \varkappa^4 a = k^2 R_f(1+\eta\sin\Omega t)\,b, \qquad \sqrt{P}\,b^{\cdot} + \varkappa^2 b = a \tag{2.3}$$

By substituting

$$a = g_1, \qquad b = lg_2, \quad t = m\tau \tag{2.4}$$

we reduce the system (2.3) to the form

$$g_1^{\cdot} + ng_1 = R(1+\eta\sin\omega\tau)g_2, \qquad g_2^{\cdot} + \frac{1}{n}g_2 = g_1 \tag{2.5}$$

where all parameters, that is l, m, n, ω and R are determined by relationships (1.11) and (1.13), just as in the case of temperature gradient modulation.

In the case of a layer with rigid boundaries we can utilize, in the method of Kantorovich, the previous approximations (1.17) and (1.18) and reduce, through substitution (2.4), the system of amplitude equations again to the form (2.5) with values of parameters determined by relationships (1.19) and (1.20).

It is easy to see that the system (2.5) is reduced to a second order equation which coincides with (1.14). In this manner the problems of stability in the case of gravity field modulation and in the case of low frequency modulation of the vertical temperature gradient, turn out to be essentially equivalent. If the solution of one of these problems is known, the solution of the other problem is obtained by a simple recalculation of parameters.

The problem of the influence of modulation of the gravity field on convective stability of fluid which fills a cavity of arbitrary shape, can also be reduced with the aid of the Kantorovich method to the integration of a system of ordinary equations of the first order with periodic coefficients. As a system of base coordinate functions we can select exact or approximate characteristic functions of the stability problem in the absence of modulation.

As an example let us examine the problem of stability of equilibrium of a fluid in a vertical circular cylinder which executes harmonic oscillations along its axis. We shall examine perturbations with the following structure:

$$v_r = v_\varphi = 0,\ v_z = v(r, \varphi),\ T = T(r, \varphi),\ p = p(z) \qquad (2.6)$$

Here r, φ and z are cylindrical coordinates. From (2.1) and (2.2) we obtain the following equations:

$$\frac{1}{\sqrt{P}}\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial z} + \Delta v + R_f(1 + \eta \sin \Omega t)\, T, \quad \sqrt{P}\,\frac{\partial T}{\partial t} - v = \Delta T \quad (2.7)$$

Here Δ is the plane Laplace operator in the variables r and φ.

On the lateral boundary of the cylinder the velocity vanishes and the following condition of heat output is fulfilled:

$$v = 0, \quad \frac{\partial T}{\partial r} = -bT \quad \text{for} \quad r = 1 \qquad (2.8)$$

Here b is the Biot number; the radius of the cylinder was chosen as the unit of length. The convective motion is assumed to be closed. This leads to the requirement for the loss across the section of the cylinder to be equal to zero.

Assuming, just as in the static case, that the basic level of instability is connected with the antisymmetric flow, the following approximations for velocity and temperature are made:

$$v = a(t) J_1(\gamma r) \cos \varphi, \quad T = b(t)\,[J_1(\gamma r) - Cr] \cos \varphi \qquad (2.9)$$

Here J_1 is the Bessel function. The boundary conditions (2.8) will be satisfied if the lower root of the equation $J_1(\gamma) = 0$ is selected as the parameter γ, i.e. $\gamma = 3.832$, and if we assume

$$C = \frac{\gamma J_1'(\gamma)}{b + 1}$$

The system of equations for amplitudes $a(t)$ and $b(t)$ obtained by the method of Kantorovich through transformations (2.4) is again reduced to the canonical system (2.5) with the following values of parameters (*):

*) The approximate critical value R_0 for the static case is quite close to the exact value. Thus, for $b = 0$ (thermally insulated boundary) and $b = \infty$ (ideally thermally conducting boundary) we obtain from (2.10) $R_0 = 71.9$ and $R_0 = 215.6$, respectively. The exact values (see [7]) are 67.9 and 215.6.

$$l = \frac{1}{\gamma^2 n}, \qquad m = \frac{n}{\gamma^2 \sqrt{P}}, \qquad n = \left[\frac{b^2 + 6b + 5 + 0.5\gamma^2}{(b+1)(b+3)} P\right]^{1/2}$$
$$R = \frac{R_f}{R_0}, \qquad R_0 = \gamma^4 \frac{b+1}{b+3} \tag{2.10}$$

3. Numerical determination of stability boundaries. The determination of regions of stability and instability of system (2.5) was carried out numerically. For this purpose the fundamental system of solutions $(g_1^{(1)}, g_2^{(1)})$ and $(g_1^{(2)}, g_2^{(2)})$ satisfying the initial conditions

$$g_i^{(k)}(0) = \delta_{ik} \tag{3.1}$$

was found on a computer by the Runge-Kutta method.

With the aid of the fundamental system we can find the "normal"solution which satisfies the condition

$$g_i(T) = \rho g_i(0) \tag{3.2}$$

where T is the period of modulation. For the factor ρ (see for example [8]) a characteristic equation

$$\rho^2 - \rho\,[g_1^{(1)}(T) + g_2^{(2)}(T)] + \exp[-(n + n^{-1})\,T] = 0 \tag{3.3}$$

is obtained in the usual manner.

The condition for the existence of periodic solutions is obtained from (3.3) for $\rho = \pm 1$

$$\pm\,[g_1^{(1)}(T) + g_2^{(2)}(T)] = 1 + \exp[-(n + n^{-1})\,T] \tag{3.4}$$

The plus and minus signs refer to "integral" and "half-integral" solutions, respectively.

The condition (3.4) ties together four parameters which enter into the system (2.5), that is R, n, ω and η. When relationship (3.4) is satisfied, the system (2.5) has a neutral periodic solution. In this manner this relationship determines the boundaries of regions of stability and instability. For a practical determination of the stability boundary it is possible, for example, to fix three parameters and to obtain the satisfaction (with prescribed accuracy) of relationship (3.4) by varying the fourth parameter. The method is readily generalized to a system of arbitrary number of equations.

Let us now turn to the presentation of numerical results. In Fig. 1, for example, a stability map is presented which corresponds to fixed values $n = 1 + \sqrt{2}$ and $R = 1.2$.

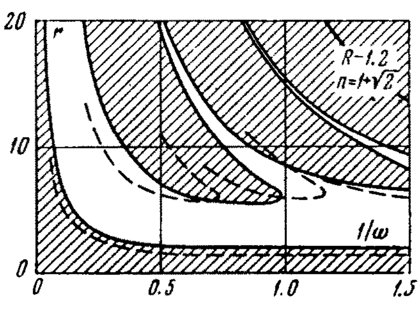

Fig. 1

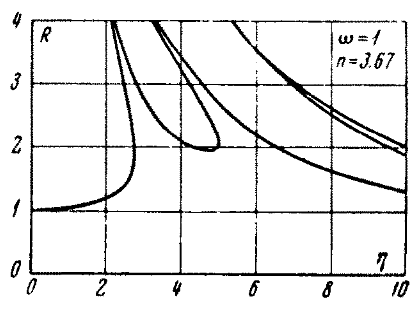

Fig. 2

For $R > 1$ in the static case the equilibrium is unstable (we recall that the reduced Rayleigh number R is determined as the ratio of the critical number R_f in the presence of modulation, to the critical number R_0 in the absence of modulation). The modulation of the parameter leads to the situation where for $R > 1$ and for fixed values of amplitude and frequency, stabilization of equilibrium occurs. In Fig. 1 regions of stability and instability are depicted in coordinates (ω^{-1}, r), where $r = R\eta$ is the absolute nondimensional amplitude of modulation (η is the relative amplitude). The map shows a fundamental band of instability adjacent to the axis $r = 0$ (the case $r = 0$ corresponds to absence of modulation, and the axis $r = 0$ belongs to the region of instability). The resonance regions of instability are located in the range of large values of r. Between the fundamental band and the resonance regions there is a band of stability (in Fig. 1 the regions of instability are shaded).

Thus, if for $R > 1$ the amplitude of modulation is increased, we can stabilize the equilibrium. Further increase in amplitude leads again to the onset of instability which is connected with the regions of resonant parametric excitation.

In the fundamental band of instability, oscillations of the integral type grow. These oscillations have a period which is equal to the period of modulation. The resonance region at the extreme left corresponds to the half-integral growing solution (the period is two times greater than the period of modulation). Subsequently regions of integral and half-integral solutions alternate.

The dashed lines in Fig. 1 show for comparison the boundaries of stability and instability regions for the case of rectangular modulation. These results are taken from paper [1]. It can be seen that the structure of regions for sinusoidal and rectangular modulation is qualitatively the same. The quantitative differences, however, are fairly significant.

In Figs. 2 and 3 numerical results are presented, they give a picture of the dependence of the critical Rayleigh number on modulation parameters. In Fig. 2 the dependence of the critical value of the reduced Rayleigh number on dimensionless amplitude of modulation η is given for fixed ω and n ($\omega = 1$, $n = 3.67$). Within the boundaries of the fundamental band of instability the Rayleigh number increases with increasing amplitude η, i.e. stabilization takes place. For $\eta > 2.7$ (for the values of ω and n as indicated) the instability is connected with the resonant parametric excitation. In this region the dependence of R and η is not monotone.

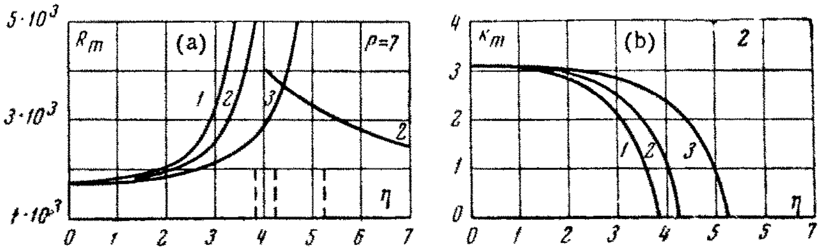

Fig. 3

Figures 3a and 3b refer to a horizontal layer with rigid boundaries. In this case the critical number R_m must be found through minimization of R_f with respect to the wave number k while the other parameters are held fixed. For fixed frequency and amplitude of modulation the parameters R_0, n and m depend on the wave number (see Eqs. (1.19) and (1.20)), and this holds consequently also for R_f. In Fig. 3, results of minimization are presented for a value of Prandtl number $P = 7$ (water). The dependence of the minimum critical Rayleigh number R_m and the critical wave number k_m on amplitude is presented for several values of nondimensional modulation frequency (curves 1–3 correspond to values of $\Omega = 10$, 50 and 100; the frequency ω which enters into the initial system of equations (2.5), in this case does not remain constant along each line because it is connected with Ω by the relationship (1.13), where m is a function of the wave number).

It is evident from Figs. 3a and 3b that the minimum critical value increases with increasing η and tends to go to infinity for some limiting η_* which increases with increasing frequency Ω. As one approaches the boundary of the fundamental band of instability ($\eta \to \eta_*$), the critical wave length increases ($k_m \to 0$).

Beyond the limits of the fundamental band ($\eta > \eta_*$) the dependence $R_m(\eta)$ is complicated. It is determined by minimization with respect to k of threshold values of R_f in resonance regions of instability (see Fig. 2). In Fig. 3, sections of curves $R_m(\eta)$ and $k_m(\eta)$ are presented for $\Omega = 50$. In the region of resonance bands the stability, in general, decreases with increasing η.

4. The limiting case of high frequencies. The numerical method permits to obtain the stability boundaries for arbitrary values of parameters of the system (1.12) or (2.5). In the limiting case of high frequencies, using the method of averaging [9], we can obtain a simple analytical equation for the stability boundary. Strictly speaking, the investigation of the high-frequency limit is physically justified only in the case of vertical oscillations of the cavity; high-frequency modulation of the temperature gradient is unavoidably connected with the appearance of a thermal skin layer which is not taken into account in the derivation of system (1.12).

In the case of rapid modulation of the gravity field, according to [9], the solution of Eq. (1.14) is presented in the form of a sum of a slowly with time varying part f_0 and a rapidly oscillating small additional component ξ

$$f(\tau) = f_0(\tau) + \xi(\tau) \tag{4.1}$$

Substituting (4.1) into the initial equation (1.14) and retaining the principal terms, we obtain after integration

$$\xi(\tau) = -R\eta\omega^{-2} f_0(\tau) \sin \omega\tau \tag{4.2}$$

(in the integration the slow part f_0 is considered to be constant). Returning to the initial equation and averaging it over the period of modulation $2\pi / \omega$, we obtain an equation for $f_0(\tau)$

$$f'_0 + (n + n^{-1}) f_0 + (1 - R + {}^1/_2 R^2\eta^2\omega^{-2}) f_0 = 0 \tag{4.3}$$

As is evident, the presence of high-frequency vibrations is in essence equivalent to renormalization of the static gravity field.

The stability boundary is determined from (4.3) (*)

$$ {}^1/_2 R^2\eta^2\omega^{-2} = R - 1 \tag{4.4}$$

Thus, in the limiting case of high frequncies the critical value of the Rayleigh number R is determined by a single parameter, the ratio of η / ω.

---

*) In the case of rectangular modulation the high frequency limit is obtained from the general characteristic relationship (2.7) of [1]. If the relative amplitude $\eta = r/R$ is introduced, then formula (3.7) of [1] is written in the form

$$ {}^1/_{12}\pi^2 R^2\eta^2\omega^{-2} = R - 1$$

This relationship differs from (4.4) only by a numerical factor.

For the averaging method to be applicable, it is necessary of course that the modulation period is small in comparison to the characteristic time of the system in the absence of modulation. In the case of heating from below this gives an inequality for the nondimensional frequency

$$\omega \gg \sqrt{R - 1 + \varepsilon^2} - \varepsilon \quad (2\varepsilon = n + n^{-1}) \tag{4.5}$$

In Fig. 4, stability boundaries are presented in the plane of parameters R and η / ω. The solid line corresponds to the asymptotic equation (4.4), obtained by the averaging method. The dashed lines represent stability boundaries which were found by means of numerical solution of system (2.5) for various values of ω and $n = 3.67$ (it is recalled that for finite ω the location of neutral lines is determined not only by the ratio η / ω, but also separately by parameters ω and n). It is evident from Fig. 4 that with increasing frequency the neutral lines which form the boundary of the fundamental region of instability, converge fairly rapidly to the limit-line determined

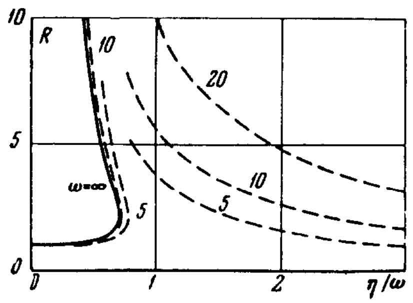

Fig. 4

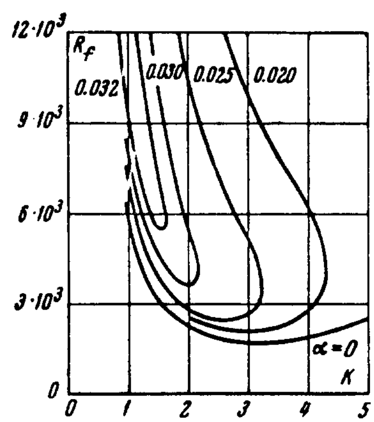

Fig. 5

by Eq. (4.4). In practice the case of high frequencies is realized even for $\omega > 10$. However, it is necessary to note that in addition to the fundamental region of instability at finite values of ω there are also resonance regions. In Fig. 4 the lower boundary of the first resonance region which corresponds to a half-integral solution is shown for $\omega = 5$, 10 and 20. With increasing frequency this boundary increases pushing out to infinity. In the high-frequency approximation only the fundamental instability region remains.

Equation (4.4) has a universal character. With its aid we can determine the boundary of the fundamental region of instability for a cavity of arbitrary form in the limiting case of high frequencies. For the transition to a concrete case it is necessary in (4.4) to separate the parameters which depend on the form of the cavity and the conditions of heating. For this purpose it is necessary in (4.4) to substitute $R = R_f / R_0$ and $\omega = mL^2\omega_0 / \sqrt{\nu\chi}$, where R_0 is the static Rayleigh number, ω_0 is the dimensional frequency of modulation. For the cases which have been examined the parameter m is determined by relationships (1.11), (1.19) and (2.10). After substitution we obtain

$$R_f = \frac{m^2 R_0}{\alpha^2}\left[1 \pm \left(1 - \frac{2\alpha^2}{m}\right)^{1/2}\right], \qquad \alpha = \frac{\omega_0 b_0 \sqrt{\nu\chi}}{gL^2} \tag{4.6}$$

Here α is a dimensionless parameter which determines the influence of high-frequency vibrations on the critical Rayleigh number (ω_0 is the frequency, b_0 is the amplitude of displacement). It is evident from the equation that absolute stabilization occurs when the parameter α reaches the value $\alpha_* = m / \sqrt{2}$.

We shall present results for a planar horizontal layer bounded by rigid isothermal planes, and for a vertical circular cylinder with general conditions of heat output.

In the case of horizontal layer, the parameters R_0 and m depend on the wave number (see (1.19) and (1.20)). In this connection Eq. (4.6) determines a family of neutral curves $R_f = R_f(k)$ which depend on the vibration parameter α (the thickness of the layer h enters into α as the characteristic dimension). These neutral curves are shown in Fig. 5. With increase in α the minimum critical value R_m increases and the critical wave number decreases. The dependence of $R_m(\alpha)$ and $k_m(\alpha)$ is presented in Figs. 6a and 6b. For $\alpha \to \alpha_*$, where $\alpha_* = 0.0347$, absolute stabilization sets in. If $\alpha > \alpha_*$, the equilibrium is stable for any value of vertical temperature gradient. It is emphasized that this applies to the limiting case of high frequencies. For high, but finite frequencies there are resonance regions of instability even for $\alpha > \alpha_*$, but they are located high (see Fig. 4). The values presented in Fig. 6 agree with results of calculations in [4]. The vibration parameter α which was introduced above is more convenient than the parameter μ of [4] because the latter contains the temperature gradient.

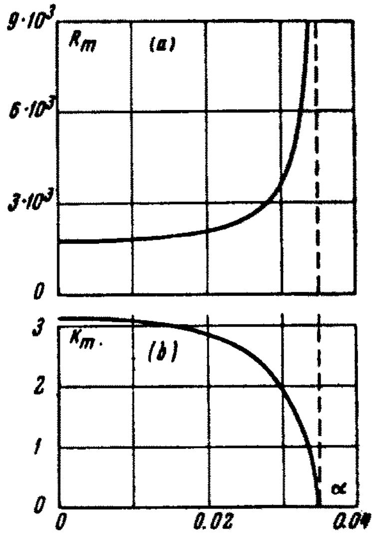

Fig. 6

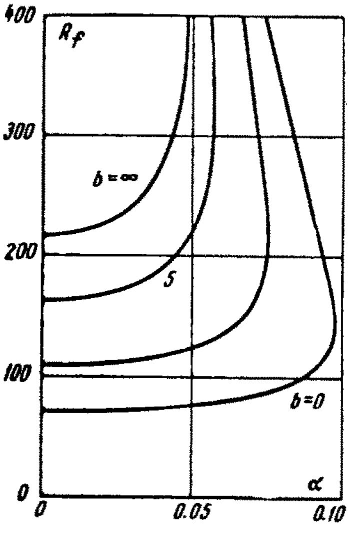

Fig. 7

In the case of the vertical cylinder, parameters R_0 and m depend on the Biot number b (see Eqs. (2.10). The dependence of R_f on the vibrational parameter α for several values of b is presented in Fig. 7 (the parameter α is determined through the radius of the cylinder). It is evident that the limiting value α_* which corresponds to absolute stabilization depends on b and decreases with increasing b. Thus, the effect of stabilization is more and more significant, the greater the heat output from the walls of the cylinder.

In conclusion we present some numerical evaluations. For complete stabilization (under conditions of high frequencies) it is necessary that the vibration parameter α reaches the limiting value α_*. The limiting value of vibrational velocity follows from this

$$\omega_0 b_0 = \alpha_* \, gL^2 (\nu\chi)^{-1/2} \qquad (4.7)$$

In order to obtain a significant effect of stabilization at values of vibrational velocity which are reasonable from the experimental point of view, one should work with fluids which have the highest possible value of parameter $\sqrt{\nu\chi}$ for sufficiently small characteristic dimensions L. Thus, for a plane layer of water ($\sqrt{\nu\chi} = 0.0038$ cm²/ s) with a thickness of 2 mm a vibrational velocity $\omega_0 b_0 = 360$ cm/s is obtained from (4.7) as necessary for complete stabilization. This means that at an amplitude of displacement of 2 mm stabilization occurs near 250 Hz. This effect is much more strongly pronounced in fluids with a high value of the parameter $\sqrt{\nu\chi}$ (glycerin, olive oil, some silicone liquids).

BIBLIOGRAPHY

1. Gershuni, G. Z. and Zhukhovitskii, E. M., On parametric excitation of convective instability. PMM Vol. 27, №5, 1963.
2. Gershuni, G. Z. and Zhukhovitskii, E. M., On convective instability of the thermal skin layer. PMTF №6, 1965.
3. Zen'kovskaia, S. M. and Simonenko, I. B., On the effect of the high frequency vibration on the onset of convection. Izv. Akad. Nauk SSSR, MZhG, №5, 1966.
4. Zen'kovskaia, S. M., Investigation of convection in a layer of fluid in the presence of vibrational forces. Izv. Akad. Nauk SSSR, MZhG, №1, 1968.
5. Venezian, G., Effect of modulation on the onset of thermal convection. J. Fluid Mech. Vol. 35, 1969.
6. Burde, G. I., Numerical investigation of convection arising in a modulated field of external forces. Izv. Akad. Nauk SSSR, MZhG, №2, 1970.
7. Ostroumov, G. A., Free Convection under Conditions of the Internal Problem. M., Gostekhizdat, 1952.
8. Coddington, E. A. and Levinson, N., Theory of Ordinary Differential Equations. McGraw-Hill, New York, 1955.
9. Landau, L. D. and Lifshits, E. M., Theoretical Physics, 2nd Ed., Vol. 1, Mechanics. M., "Nauka", 1965.

Translated by B. D.

HYPERSONIC FLOW PAST A DELTA WING

PMM Vol. 34, №3, 1970, pp. 481-490

A. L. GONOR
(Moscow)

(Received January 31, 1969)

The mode of flow over the windward side of a supersonic leading edge wing is examined. In spite of a number of investigations [1–4], this problem has not been solved correctly. The difficulty consists in the fact that in the flow field behind a strong wave there are regions of homogeneous potential and vortex flows which must be matched with sufficient smoothness.

An analytical theory is developed below for hypersonic flow past a wing with an attached wave. This theory allows to carry out the necessary conjuction of flows.